

# Lagrangian Actions on Elliptical Solutions of 2-Body and 3-Body Problems with Fixed Energies<sup>\*</sup>

Ying Lv and Shiqing Zhang

College of Mathematics and Statistics, Southwestern University, Chongqing 400715, P.R.China

Yangtze Center of Mathematics, Sichuan University, Chengdu 610064, P.R.China

## Abstract

Based on the works of Gordon ([4]) and Zhang-Zhou([8])) on the variational minimizing properties for Keplerian orbits and Lagrangian solutions of Newtonian 2-body and 3-body problems, we use the constrained variational principle of Ambrosetti-Coti Zelati ([1]) to compute the Lagrangian actions on Keplerian and Lagrangian elliptical solutions with fixed energies, we also find an interesting relationship between period and energy for Lagrangian elliptical solutions with Newtonian potentials.

**Key Words:** 2 and 3-body problems, Keplerian orbits, Lagrangian solutions, Fixed energy, Lagrangian actions.

**2000MSC, 70G75, 70F07, 70F10.**

## 1 Introduction and Main Results

In [4], Gordon proved that the Keplerian orbits minimize the Lagrangian action of the Keplerian 2-body problems with a fixed period, in [8], Zhang-Zhou generalized the result of Gordon to Newtonian 3-body problems and proved that the Lagrangian elliptical orbits with equilateral configurations minimize the Lagrangian action with a fixed period.

In this note, we try to generalize the above cases for the fixed period to the fixed energy.

Consider Keplerian two-body problem with a fixed energy  $h < 0$ :

$$\begin{cases} \ddot{x}(t) + \nabla V(x) = 0, & x \in R^2 \\ \frac{1}{2}|\dot{x}|^2 + V(x) = h, \end{cases} \quad (1)$$

where

$$V(x) = \frac{-a}{|x|}, \quad a > 0 \quad (2)$$

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Let  $W^{1,2}(R/Z, R^2)$  denote the Sobolev space with period 1 and the usual inner product and norm:

$$\langle u, v \rangle = \int_0^1 (u \cdot v + \dot{u} \cdot \dot{v}) dt \quad (3)$$

$$\|u\| = \langle u, u \rangle^{1/2} \quad (4)$$

For two-body problems with a fixed energy  $h$ , Ambrozeth-CotiZelati ([1]) defined Lagrangian action:

$$f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \cdot \int_0^1 (h - V(u)) dt \quad (5)$$

and the following constrained manifold:

$$M_h = \left\{ u \in W^{1,2} \mid u(t) \not\equiv 0 \left| \int_0^1 \left( \frac{1}{2} V'(u) u + V(u) \right) dt = h \right. \right\} \quad (6)$$

and they proved that the critical point  $\tilde{u}$  of  $f(u)$  on  $M_h$  corresponds to a noncollision  $T$ -periodical solution  $\tilde{q}(t) = \tilde{u}(t/T)$  of the system (1) after a scaling for the period  $T$ :

$$\begin{aligned} \frac{1}{T^2} &= \frac{\int_0^1 V'(\tilde{u}) \cdot \tilde{u} dt}{\int_0^1 |\dot{\tilde{u}}|^2 dt} \\ &= \frac{\int_0^1 (h - V(\tilde{u})) dt}{\frac{1}{2} \int_0^1 |\dot{\tilde{u}}|^2 dt} \end{aligned} \quad (7)$$

For  $N$ -body type problems, they also showed the similar variational principle.

For two-body problem (1), we have the following Theorem:

**Theorem 1.1** Let  $\deg u$  denote the winding number of the loop  $u$  respect to the origin, define

$$\Lambda_1 = \{u \in M_h, \deg u \neq 0\}. \quad (8)$$

Then the global minimum of  $f(u)$  on the closure  $\overline{\Lambda_1}$  exists and equals to  $\frac{9}{16} \cdot 2^{-1/3} (\pi a)^2 (-h)^{-1}$ , and the minimizer  $\tilde{u}(t)$  of  $f(u)$  on  $\overline{\Lambda_1}$  are exactly corresponding to the stright line collision solution  $\tilde{x}(t) = u(t/T)$  or Keplerian elliptical solution  $x(t) = \tilde{u}(t/T)$  under a scaling transform:

$$T = 2\pi(-2h)^{-3/2}a \quad (9)$$

and  $x(t)$  has energy  $h$ .

For Newtonian 3-body problems with a fixed energy  $E$ :

$$\begin{cases} m_i \ddot{q}_i = \frac{-\partial V(q)}{\partial q_i}, \\ \frac{1}{2} \sum_{i=1}^3 m_i |\dot{q}_i|^2 + V(q) = E, \end{cases} \quad (10)$$

where

$$V(q) = - \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|}. \quad (11)$$

We define.

$$F(u) = \frac{1}{2} \int_0^1 \sum_{i=1}^3 m_i |\dot{u}_i|^2 dt \cdot \int_0^1 (E - V(u)) dt, \quad (12)$$

$$u \in \Lambda_2 = \left\{ \begin{array}{l} u = (u_1, u_2, u_3) | u_i \in W^{1,2}, \sum_{i=1}^3 m_i u_i = 0, \\ \deg(u_i - u_j) \neq 0, \\ \int_0^1 (V(u) + \frac{1}{2} V'(u)u) dt = E \end{array} \right\} \quad (13)$$

Then we have

**Theorem 1.2** The global minimizers of  $f(u)$  on  $\Lambda_2$  are just the Lagrangian elliptical solutions and the period for the elliptical orbits is

$$T = 2\pi \cdot \left( \frac{\sum_{1 \leq i < j \leq 3} m_i m_j}{-2E} \right)^{3/2} \quad (14)$$

and the Lagrangian action is

$$2^{-13/3} (3\pi)^2 \left( \sum_{1 \leq i < j \leq 3} m_i m_j \right)^3 \cdot (-E)^{-1}, \quad (15)$$

## 2 The Proof of Theorem 1.1

**Lemma 2.1**(Newton [6]) For Keplerian elliptical orbits of two-body problem (1), the period  $T$  and energy  $h$  has the following relationship:

$$T = 2\pi(-2h)^{-3/2}a \quad (16)$$

**Lemma 2.2**(Gordon [4]) Let  $\bar{\Lambda}$  be the  $W^{1,2}(R/TZ, R^2)$  completion of the following loop space with period  $T$ :

$$\Lambda = \{x(t) \in C^\infty(R/TZ, R^2) | x(t) \neq 0, \deg x \neq 0\} \quad (17)$$

We define the Lagrangian action:

$$g(x) = \int_0^T \left( \frac{1}{2} |\dot{x}|^2 + \frac{a}{|x|} \right) dt \quad (18)$$

Then the minimizers of  $g(x)$  on  $\bar{\Lambda}$  are the Keplerian elliptical solutions or the straight line collision solution with one leg, and the minimum is

$$(3\pi) \left( \frac{T}{2\pi} \right)^{1/3} \cdot a^{2/3} = \frac{3}{2} (2\pi)^{2/3} a^{2/3} T^{1/3}. \quad (19)$$

**Lemma 2.3**([3]) Let  $u(t)$  be a critical point of  $f(u)$  on  $\bar{\Lambda}$  and let  $x(t) = u(t/T)$ , then

$$\begin{aligned} [4f(u)]^{1/2} &= \int_0^T \left[ \frac{1}{2} |\dot{x}|^2 + \frac{1}{2} V'(x)x \right] dt \\ &= \int_0^T \left[ \frac{1}{2} |\dot{x}|^2 + \frac{a}{2} \frac{1}{|x|} \right] dt \end{aligned} \quad (20)$$

Now we can prove Theorem 1.1:

By Lemmas 2.1-2.3, we have

$$\begin{aligned} [4f(u)]^{1/2} &\geq \frac{3}{2} (2\pi)^{2/3} \left( \frac{a}{2} \right)^{2/3} T^{1/3} \\ &= \frac{3}{2} \pi^{2/3} a^{2/3} (2\pi)^{1/3} (-2h)^{-1/2} a^{1/3} \end{aligned} \quad (21)$$

$$f(u) \geq \frac{9}{16} 2^{-1/3} (\pi a)^2 (-h)^{-1} \quad (22)$$

and  $f(u)$  attains the infimum if and only if the minimizers are Keplerian elliptical orbits or the collision solution with one leg.

### 3 The Proof of Theorem 1.2

**Lemma 3.1** For a Lagrangian elliptical solution ([5])  $q = (q_1, q_2, q_3)$  with period  $T$ , the energy  $E$  for masses  $m_1, m_2, m_3$  is

$$E = \left( -\frac{1}{2} \right) \left( \frac{T}{2\pi} \right)^{-2/3} \left( \sum_{1 \leq i < j \leq 3} m_i m_j \right). \quad (23)$$

**Proof.** The Lagrangian solution ([5]) is

$$q(t) = x(t)(\alpha_1, \alpha_2, \alpha_3), \quad (24)$$

where  $|\alpha_1 - \alpha_2| = |\alpha_2 - \alpha_3| = |\alpha_3 - \alpha_1| = 1$ ,  $x(t)$  is the Keplerian elliptical orbit satisfying

$$\ddot{x}(t) = \frac{-ax(t)}{|x(t)|^3} \quad (25)$$

From (24), we have

$$q_i(t) - q_j(t) = x(t)(\alpha_i - \alpha_j), \quad (26)$$

$$\dot{q}_i(t) - \dot{q}_j(t) = \dot{x}(t)(\alpha_i - \alpha_j), \quad (27)$$

$$\begin{aligned} &\frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 - \frac{M}{|q_i - q_j|}, \\ &= \frac{1}{2} |\dot{x}|^2 - \frac{M}{|x|} \triangleq h \end{aligned} \quad (28)$$

where  $M = \sum_{i=1}^3 m_i$ .

We notice that the energy for the Lagrangian elliptical solutions is

$$\begin{aligned}
E &= \frac{1}{2} \sum m_i |\dot{q}_i|^2 - \sum_{i < j} \frac{m_i m_j}{|q_i - q_j|} \\
&= \frac{1}{M} \sum_{i < j} m_i m_j \left[ \frac{|\dot{q}_i - \dot{q}_j|^2}{2} - \frac{M}{|q_i - q_j|} \right] \\
&= \frac{1}{M} \sum_{i < j} m_i m_j h
\end{aligned} \tag{29}$$

For Keplerian orbits  $(q_i - q_j)$ , we use Lemma 2.1 to get

$$T = 2\pi(-2h)^{-3/2} \cdot M, \tag{30}$$

$$\left( \frac{T}{2\pi M} \right)^{-2/3} = -2h \tag{31}$$

Hence

$$E = \left( \sum_{i < j} m_i m_j \right) \left( -\frac{1}{2} \right) \cdot \left( \frac{T}{2\pi M} \right)^{-2/3}, \tag{32}$$

$$T = 2\pi \left( \frac{\sum_{i < j} m_i m_j}{-2E} \right)^{3/2} \cdot M \tag{33}$$

**Lemma 3.2**([7]) Let  $u = (u_1, u_2, u_3)$  be a critical point of  $F(u)$  on  $\Lambda_2$ , let  $q(t) = u(t/T)$ , then

$$E = \frac{1}{2} \sum m_i |\dot{q}_i|^2 - \sum_{i < j} \frac{m_i m_j}{|q_i - q_j|} \tag{34}$$

$$[4F(u)]^{1/2} = \int_0^T \left[ \frac{1}{2} \sum_{i=1}^3 m_i |\dot{q}_i|^2 + \frac{1}{2} V'(q) \cdot q \right] dt \tag{35}$$

$$= \int_0^T \left[ \frac{1}{2} \sum_{i=1}^3 m_i |\dot{q}_i|^2 - \frac{1}{2} V(q) \right] dt \tag{36}$$

Similar to [8], we have

$$\sum_i m_i |\dot{q}_i|^2 = \frac{1}{M} \sum_{i < j} m_i m_j |\dot{q}_i - \dot{q}_j|^2, \tag{37}$$

Hence

$$[4F(u)]^{1/2} = \int_0^T \frac{1}{M} \sum_{i < j} m_i m_j \left[ \frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 + \frac{M}{2} \frac{1}{|q_i - q_j|} \right] \quad (38)$$

By Gordon's Lemma 2.2,

$$[4F(u)]^{1/2} \geq \frac{1}{M} \left( \sum_{i < j} m_i m_j \right) \cdot \left[ \frac{3}{2} (2\pi)^{2/3} \left( \frac{M}{2} \right)^{2/3} T^{1/3} \right] \quad (39)$$

and  $[4F(u)]^{1/2}$  attains the infimum if and only if for  $1 \leq i \neq j \leq 3$ ,

$$\int_0^T \left[ \frac{1}{2} |\dot{q}_i - \dot{q}_j|^2 + \frac{M}{2} \frac{1}{|q_i - q_j|} \right] dt = \frac{3}{2} (2\pi)^{2/3} \left( \frac{M}{2} \right)^{2/3} T^{1/3} \quad (40)$$

Then similar to the proof in Zhang-Zhou [8], the equations (39) hold if and only if  $q = (q_1, q_2, q_3)$  is a Lagrangian elliptical solution, so we know the minimizers of  $[4F(u)]^{1/2}$  correspond to Lagrangian elliptical solutions after a scaling.

By Lemma 3.1, we have

$$\begin{aligned} [4F(u)]^{1/2} &\geq \frac{3}{2} (2\pi)^{2/3} \left( \frac{1}{2} \right)^{2/3} M^{-1/3} \left( \sum_{i < j} m_i m_j \right) \cdot (2\pi)^{1/3} \cdot \left( \frac{\sum_{i < j} m_i m_j}{-2E} \right)^{1/2} \cdot M^{1/3} \\ &= \frac{3}{2} (2\pi) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{1/6} \cdot \left( \sum_{i < j} m_i m_j \right)^{3/2} \cdot (-E)^{-1/2} \end{aligned} \quad (41)$$

$$F(u) \geq 2^{-\frac{13}{3}} \cdot (3\pi)^2 \cdot \left( \sum_{i < j} m_i m_j \right)^3 \cdot (-E)^{-1} \quad (42)$$

From the above proof, we know that  $F(u)$  attain the infimum on  $\Lambda_2$  if and only if the minimizers correspond Lagrangian elliptical solutions after a scaling and the Lagrangian action on Lagrangian elliptical solutions has the value in (15).

## References

- [1] Ambrosetti A. and Coti Zelati V., Closed orbits of fixed energy for singular Hamiltonian systems, Arch. Rational Mech. Anal. 112(1990), 339-362.
- [2] Ambrosetti A. and Coti Zelati, Periodic solutions of singular Lagrangian systems, Birkhäuser, 1993.
- [3] Bessi U., Multiple closed orbits of fixed energy for gravitational potentials, J. Diff. Equ. 104(1993), 1-10.
- [4] Gordon, W., A minimizing property of Keplerian orbits, American J. Math. 99(1977), 961-971.
- [5] Lagrange J., Essai sur le probleme des trois corps, 1772, Ouvres 3, 1873, 229-331.
- [6] Newton I., Principia Mathematica Philosophiae Naturalis, 1687.
- [7] Zhang S., Multiple closed orbits of fixed energy for  $N$ -body-type problems with gravitatioal potentials, J. Math. Anal. Appl. 208(1997), 462-475.
- [8] Zhang S.Q. and Zhou Q., A minimizing property of Lagrangian solutions, Acta Math. Sinica, English Series 17(2001), 497-500.